4-4 Notes: Fundamental Theorem of Calculus Name ______________________
Calculus AB

The Fundamental Theorem of Calculus I

If a function f is continuous on the closed interval $[a,b]$ and F is an antiderivative of f on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Guidelines for Using the Fundamental Theorem of Calculus

1) Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2) When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,dx = \left. F(x) \right]_a^b = F(b) - F(a)$$

For instance, to evaluate $\displaystyle\int_1^3 x^3\,dx = \left.\frac{x^4}{4}\right]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20$

3) It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \left. F(x) + C \right]_a^b = \left[F(b)+C\right] - \left[F(a)+C\right] = F(b) - F(a)$$

Example 1) Evaluate the definite integral: $\displaystyle\int_2^5 (-3x+4)\,dx$

Example 2) Evaluate the definite integral: $\displaystyle\int_{-1}^1 (x^3 - 9x)\,dx$

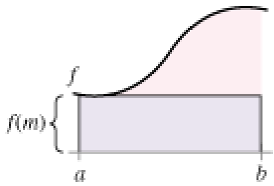
Example 3) Evaluate the definite integral: $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}} dx$

Example 4) Evaluate the definite integral: $\int_{0}^{4} \left| x^2 - 4x + 3 \right| dx$

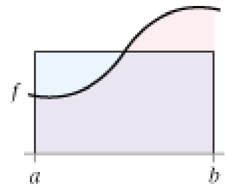
Example 5) Find the area of the region bounded by the graphs of the equations $y = -x^2 + 3x$ and y = 0

Mean Value Theorem for Integrals

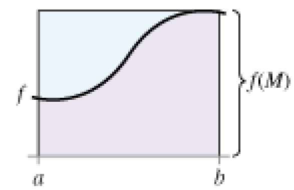
If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$.



Inscribed rectangle
(less than actual area)



Mean Value Rectangle
(equal to actual area)



Circumscribed Rectangle
(greater than actual area)

$$\int_a^b f(m)\,dx = f(m)(b-a)$$

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$$\int_a^b f(M)\,dx = f(M)(b-a)$$

$$f(m)(b-a) < \int_a^b f(x)\,dx$$

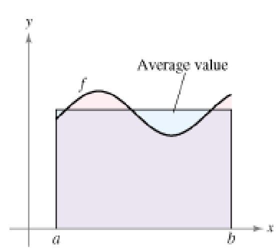
$$f(M)(b-a) > \int_a^b f(x)\,dx$$

- In other words, somewhere between the inscribed rectangle and the circumscribed rectangle, there is a rectangle whose area is precisely the area under the curve

Example 6) Find the value of c guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = \cos x$ over the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

Average Value of a Function

If f is integrable on the closed interval $[a,b]$, then the **average value** of f on the interval is $\frac{1}{b-a}\int_a^b f(x)\,dx$.



- This is a result of the Mean Value Theorem of Integrals. If $\int_a^b f(x)\,dx = f(c)(b-a)$, then $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.
- $f(c)$ from the Mean Value Theorem is called the average value of f from $[a,b]$.

Example 7) $f(x) = \dfrac{x^2+1}{x^2}$ $\left[\dfrac{1}{2}, 2\right]$

a) Find the average value of the function over the interval.

b) Find all values of x in the interval for which the function equals its average value.

Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

- In other words, where the integrand is continuous, the derivative of a definite integral with respect to its upper limit is equal to the integrand evaluated at the upper limit

Example 8) Use the Second Fundamental Theorem of Calculus to find $F'(x)$.

a) $F(x) = \int_1^x \sqrt[4]{t}\, dt$

b) $F(x) = \int_2^{x^2} \frac{1}{t^3}\, dt$

c) $F(x) = \int_0^{x^2} \sin(\theta^2)\, d\theta$